APPENDIX

We first should find $\frac{\partial H(t)}{\partial x_{i\mu}(t)}$ such that $\frac{\partial Q_{i\kappa}(t)}{\partial x_{i\mu}(t)} = 0$ in hard membership state. Having equations (10), (13), and (14) in the main text, the term $\frac{\partial H(t)}{\partial x_{i\mu}(t)}$ is calculated accordingly. Using Eq.(47) in the main text, we have

$$\begin{aligned}\frac{\partial H(t)}{\partial x_{i\mu}(t)} &= \frac{1}{2}\gamma_1 \frac{\partial}{\partial x_{i\mu}(t)} \sum_{\kappa} \{\log(\frac{p_{\kappa}^{in}}{p_{\kappa}^{out}}) \sum_{a} \sum_{b \neq a} (A_{ab} x_{a\kappa}(t) x_{b\kappa}(t))\} \\ &+ \frac{1}{2}\gamma_2 \frac{\partial}{\partial x_{i\mu}(t)} \sum_{\kappa} \{\log(\frac{1-p_{\kappa}^{in}}{1-p_{\kappa}^{out}}) \sum_{a} \sum_{b \neq a} (A_{ab} x_{a\kappa}(t) x_{b\kappa}(t))\} \\ &+ \frac{1}{2} \frac{\partial}{\partial x_{i\mu}(t)} \sum_{\kappa} \{\log(p_{\kappa}^{out}) \sum_{a} \sum_{b \neq a} (A_{ab} x_{a\kappa}(t))\} + \frac{1}{2} \frac{\partial}{\partial x_{i\mu}(t)} \sum_{\kappa} \{n_{\kappa} \log(\frac{n_{\kappa}}{n})\}.\end{aligned} \tag{1}$$

Some terms are calculated mathematically in the following:

$$\sum_{a} \sum_{b \neq a} (A_{ab} x_{a\mu}(t)) = \sum_{a} \sum_{b} (A_{ab} x_{a\mu}(t)) = \sum_{b} \sum_{a} (A_{ab} x_{a\mu}(t)) = \sum_{b} k_{b\mu}, \tag{2}$$

$$\begin{aligned}\sum_{a} \sum_{b \neq a} (J_{ab} x_{a\mu}(t) x_{b\mu}(t)) =& \sum_{a} \sum_{b \neq a} [x_{a\mu}(t) x_{b\mu}(t)(1 - A_{ab})] \\ =& \sum_{a} \sum_{b \neq a} [x_{a\mu}(t) x_{b\mu}(t)] - \sum_{a} \sum_{b \neq a} [x_{a\mu}(t) x_{b\mu}(t) A_{ab}] \\ =& \frac{2l_{\mu}^{in}}{p_{\mu}^{in}} - 2l_{\mu}^{in} = \frac{2l_{\mu}^{in}(1 - p_{\mu}^{in})}{p_{\mu}^{in}},\end{aligned} \tag{3}$$

$$\begin{aligned}\sum_{a \neq i} (J_{ai} x_{a\mu}(t)) =& \sum_{a \neq i} [x_{a\mu}(t)(1 - A_{ab})] = (\sum_{a \neq i} x_{a\mu}(t)) - \sum_{a \neq i} (x_{a\mu}(t) A_{ab}) \\ =& n_{\mu} - k_{i\mu} - x_{i\mu}(t),\end{aligned} \tag{4}$$

$$\sum_{a \neq i} J_{ai} = \sum_{a \neq i} (1 - A_{ab}) = n - k_i - 1, \tag{5}$$

$$\begin{aligned}\frac{\partial p_{\mu}^{in}}{\partial x_{i\mu}(t)} =& \frac{(2\sum_{a \neq i} x_{a\mu} A_{ia})(2l_{\mu}^{in}/p_{\mu}^{in}) - 4(n_{\mu} - x_{i\mu}(t)) l_{\mu}^{in}}{(2l_{\mu}^{in}/p_{\mu}^{in})^2} \\ =& \frac{k_{i\mu} p_{\mu}^{in} - (p_{\mu}^{in})^2 (n_{\mu} - x_{i\mu}(t))}{l_{\mu}^{in}},\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{\partial p_{\mu}^{out}}{\partial x_{i\mu}(t)} =& \frac{(\sum_{a \neq i} \{(1 - 2x_{a\mu}(t)) A_{ia}\})(l_{\mu}^{out}/p_{\mu}^{out}) - l_{\mu}^{out}(\sum_{a \neq i}(1 - 2x_{a\mu}(t)))}{(l_{\mu}^{out}/p_{\mu}^{out})^2} \\ =& \frac{(k_i - 2k_{i\mu}) p_{\mu}^{out} - (p_{\mu}^{out})^2 ((n-1) - 2(n_{\mu} - x_{i\mu}(t)))}{l_{\mu}^{out}},\end{aligned} \tag{7}$$

therefore,

$$\begin{aligned}\frac{\partial}{\partial x_{i\mu}(t)}\log(\frac{p_\mu^{in}}{p_\mu^{out}}) =&\frac{\partial}{\partial x_{i\mu}(t)}\log(p_\mu^{in}) - \frac{\partial}{\partial x_{i\mu}(t)}\log(p_\mu^{out}) = \frac{(\frac{\partial p_\mu^{in}}{\partial x_{i\mu}(t)})}{p_\mu^{in}} - \frac{(\frac{\partial p_\mu^{out}}{\partial x_{i\mu}(t)})}{p_\mu^{out}}\\ =&\frac{k_{i\mu} - p_\mu^{in}(n_\mu - x_{i\mu}(t))}{l_\mu^{in}} - \frac{(k_i - 2k_{i\mu}) - p_\mu^{out}((-1) - 2(n_\mu - x_{i\mu}(t)))}{l_\mu^{out}},\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{\partial}{\partial x_{i\mu}(t)}\log(\frac{1-p_\mu^{in}}{1-p_\mu^{out}}) =&\frac{\partial}{\partial x_{i\mu}(t)}\log(1-p_\mu^{in}) - \frac{\partial}{\partial x_{i\mu}(t)}\log(1-p_\mu^{out}) = \frac{(\frac{\partial p_\mu^{out}}{\partial x_{i\mu}(t)})}{1-p_\mu^{out}} - \frac{(\frac{\partial p_\mu^{in}}{\partial x_{i\mu}(t)})}{1-p_\mu^{in}}\\ =&p_\mu^{out}\frac{(k_i - 2k_{i\mu}) - p^{out}((n-1) - 2(n_\mu - x_{i\mu}(t)))}{l_\mu^{out}(1-p_\mu^{out})} - p_\mu^{in}\frac{k_{i\mu} - p_\mu^{in}(n_\mu - x_{i\mu}(t))}{l_\mu^{in}(1-p_\mu^{in})},\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{\partial}{\partial x_{i\mu}(t)}\log(p_\mu^{out}) =& \frac{\frac{\partial p_\mu^{out}}{\partial x_{i\mu}(t)}}{p_\mu^{out}}\\ =&\frac{(k_i - 2k_{i\mu}) - p_\mu^{out}((n-1) - 2(n_\mu - x_{i\mu}(t)))}{l_\mu^{out}},\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{\partial}{\partial x_{i\mu}(t)}\log(1-p_\mu^{out}) =& -\frac{\frac{\partial p_\mu^{out}}{\partial x_{i\mu}(t)}}{1-p_\mu^{out}}\\ =& -p_\mu^{out}\frac{(k_i - 2k_{i\mu}) - p_\mu^{out}((n-1) - 2(n_\mu - x_{i\mu}(t)))}{l_\mu^{out}(1-p_\mu^{out})}.\end{aligned} \tag{11}$$

Then using Eq.(1), we have

$$\begin{aligned}\frac{\partial H(t)}{\partial x_{i\mu}(t)} =&\frac{1}{2}\gamma_1\{(\sum_a\sum_{b\neq a}(A_{ab}x_{a\mu}(t)x_{b\mu}(t)))\frac{\partial(\log(\frac{p_\mu^{in}}{p_\mu^{out}}))}{\partial x_{i\mu}(t)}\\ &+(2\sum_{a\neq i}(A_{ai}x_{a\mu}(t)))\log(\frac{p_\mu^{in}}{p_\mu^{out}})\} + \frac{1}{2}\gamma_2\{(\sum_a\sum_{b\neq a}(J_{ab}x_{a\mu}(t)x_{b\mu}(t)))\frac{\partial(\log(\frac{1-p_\mu^{in}}{1-p_\mu^{out}}))}{\partial x_{i\mu}(t)}\\ &+(2\sum_{a\neq i}(J_{ai}x_{a\mu}(t)))\log(\frac{1-p_\mu^{in}}{1-p_\mu^{out}})\} + \frac{1}{2}\{(\sum_a\sum_{b\neq a}(A_{ab}x_{a\mu}(t)))\frac{\partial(\log(p_\mu^{out}))}{\partial x_{i\mu}(t)} + (\sum_{a\neq i}A_{ai})\log(p_\mu^{out})\}\\ &+\frac{1}{2}\{(\sum_a\sum_{b\neq a}(J_{ab}x_{a\mu}(t)))\frac{\partial(\log(1-p_\mu^{in}))}{\partial x_{i\mu}(t)} + (\sum_{a\neq i}J_{ai})\log(1-p_\mu^{in})\} + \frac{\partial}{x_{i\mu}(t)}\sum_\mu(n_\mu\log(\frac{n_\mu}{n})).\end{aligned} \tag{12}$$